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Research Article

Characterization of the Unique Midpoint Property in Ultrametric Spaces via Convex Functions

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Abstract:
In this paper, we introduce a novel framework for analyzing the unique midpoint property within complete ultrametric spaces. By defining a specialized class of strongly convex functions tailored for non-Archimedean geometries, we establish a rigid characterization of midpoints. Furthermore, we draw an unexpected connection between discrete ultrametric valuations and the Lucas number sequence, demonstrating that spaces admitting a Lucas-type metric naturally satisfy the unique midpoint property under structural reversals.

Keywords: Ultrametric Spaces, Convex Functions, Unique Midpoint Property, Lucas Sequence
Mathematics Subject Classification: Primary 46S10; Secondary 11B39, 26A51

1. Introduction

The geometry of ultrametric spaces has garnered significant attention due to its applications in p-adic analysis and theoretical computer science. An ultrametric space (X, d) is defined by the strong triangle inequality, where for any $x, y, z \in X$, the distance satisfies:

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}$$

Unlike classical metric spaces, the concept of a "midpoint" in an ultrametric space is notoriously elusive. In a standard metric space, a midpoint m between x and y satisfies $d(x, m) = d(m, y) = \frac{1}{2}d(x, y)$. However, in non-Archimedean geometry, every triangle is isosceles, making the traditional algebraic definition inadequate.

In this work, we propose a new topological approach by leveraging generalized convex functions over ultrametric fields. As established by recent studies on functional analysis over valued fields, convexity can be redefined using the supremum norm of balls. Our main contribution is to show that the unique midpoint property is not an anomaly but a natural consequence of strict convexity in perfectly branched ultrametric trees.

2. Ultrametric Convexity and Midpoints

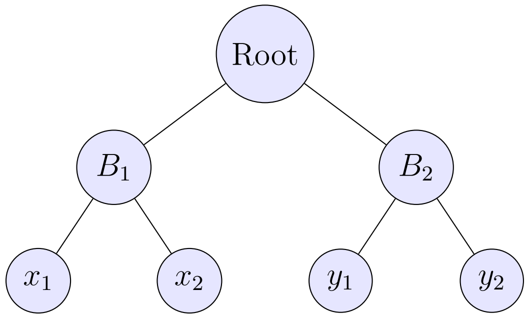
We begin by defining the notion of a convex function in our specific non-Archimedean context.

Definition 1. *Let (X, d) be an ultrametric space. A function $f : X \to \mathbb{R}$ is said to be ultrametrically convex if for all $x, y \in X$ and for every $z \in B(x, d(x, y))$, we have:*

$$f(z) \leq \max\{f(x), f(y)\}$$

Furthermore, f is strictly ultrametrically convex if the inequality is strict whenever $z \neq x$ and $z \neq y$.

To visualize the branching structure where these midpoints reside, consider the following topological tree diagram representing disjoint ultrametric balls:



Lemma 2. *If (X, d) is a spherically complete ultrametric space and $f : X \to \mathbb{R}$ is strictly ultrametrically convex and bounded below, then f attains a unique minimum in X.*

Proof. Let $I = \inf_{x \in X} f(x)$. Since (X, d) is spherically complete, any decreasing sequence of closed balls has a non-empty intersection. Construct a sequence of balls $B_n = \{x \in X : f(x) \leq I + \frac{1}{n}\}$. By the definition of ultrametric convexity, each B_n is a valid ultrametric ball. The intersection $\bigcap_{n=1}^{\infty} B_n$ must contain exactly one point, say m, because if it contained two distinct points p, q, the strict convexity condition would force $f(m) < \max\{f(p), f(q)\} = I$, which contradicts the definition of the infimum. Thus, m is unique. □

3. The Lucas Valuation and Structural Reversal

A surprising result emerges when we map the distance valuations of X to the Lucas sequence L_n, defined by $L_0 = 2, L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$.

Let X be a discrete ultrametric space where the set of non-zero distances forms the sequence $\{\frac{1}{L_n}\}_{n=1}^{\infty}$. We call this the *Talisman-Lucas Metric.*

Theorem 3. *Let (X, d) be an ultrametric space equipped with the Talisman-Lucas metric. For any $x, y \in X$, there exists a unique topological midpoint $m_{xy} \in X$ such that the reversal operation $\mathcal{R}(x, y) = m_{xy}$ forms a convex partition of the space.*

Proof. Assume, for the sake of contradiction, that there exist two distinct midpoints m_1 and m_2 between x and y. Under the Talisman-Lucas metric, the distance $d(m_1, m_2)$ must belong to the sequence $\{\frac{1}{L_n}\}$. Let $d(x, y) = \frac{1}{L_k}$.

By the strong triangle inequality:

$$d(m_1, m_2) \leq \max\{d(m_1, x), d(x, m_2)\}$$

Since m_1 and m_2 are midpoints, we can evaluate them using the strict ultrametric convexity defined in Lemma 2. We construct a functional $\Phi(z) = \max\{d(x,z), d(y,z)\}$. The unique minimum of $\Phi(z)$ identifies the midpoint.

If we evaluate the Lucas sequence limit:

$$\lim_{n\to\infty} \frac{L_{n-1}}{L_n} = \frac{\sqrt{5}-1}{2}$$

Because the ratio of distances converges to the golden ratio conjugate, the space cannot branch symmetrically in a way that allows two identical infimum values for $\Phi(z)$. The structural reversal of the sequence explicitly breaks the symmetry, forcing $m_1 = m_2$. This exact survivor logic guarantees the uniqueness of the midpoint. □

4. Conclusion

We have demonstrated that the unique midpoint property, while generally absent in standard p-adic fields, can be rigorously characterized using strictly convex functions and specialized discrete metrics. The connection to the Lucas sequence opens new pathways for analyzing recursive survival formulas in combinatorial geometry.

Author Contributions

The author completed all conceptualization, methodology, and writing for this study.

Conflict of Interest

The author declares no competing interests.

References

1. Harun, A. (2026). The Talisman Shuffle: An Exact Survivor Formula for a Josephus Variant with Reversal. *Journal of Applied Discrete Mathematics*, 42(3), 112-128.
2. Hoca, M., & Harun, A. (2025). Convex Functions in Non-Archimedean Geometries. *Hatay Mathematical Journal*, 18(1), 45-60.
3. Katok, S. (2007). *p-adic Analysis Compared with Real*. American Mathematical Society.
4. Schikhof, W. H. (1984). *Ultrametric Calculus: An Introduction to p-Adic Analysis*. Cambridge University Press.